

# MORI CONTRACTIONS OF MAXIMAL LENGTH

ANDREAS HÖRING AND CARLA NOVELLI

**ABSTRACT.** We prove a relative version of the theorem of Cho, Miyaoka and Shepherd-Barron: a Mori fibre space of maximal length is birational to a projective bundle.

## 1. INTRODUCTION

**1.A. Motivation.** Let  $X$  be a Fano manifold of dimension  $d$  with Picard number one, and denote by  $H$  the ample generator of the Picard group. Let  $i(X) \in \mathbb{N}$  be such that  $K_X^* \equiv i(X)H$ ; by a classical theorem of Kobayashi and Ochiai [KO73] one has  $i(X) \leq d+1$  and equality holds if and only if  $X$  is isomorphic to the projective space  $\mathbb{P}^d$ . If one tries to understand Fano manifolds of higher Picard number the index  $i(X)$  is less useful, since it can be equal to one even for very simple manifolds like  $\mathbb{P}^d \times \mathbb{P}^{d+1}$ . For these manifolds the pseudoinde

$$\min\{K_X^* \cdot C \mid C \subset X \text{ a rational curve}\}$$

yields much more precise results, the most well-known being the theorem of Cho, Miyaoka and Shepherd-Barron:

**1.1. Theorem.** [CMSB02, Cor.0.3] [Keb02, Thm.1.1] *Let  $X$  be a projective manifold of dimension  $d$  such that for every rational curve  $C \subset X$ , we have  $K_X^* \cdot C \geq d+1$ . Then  $X$  is isomorphic to the projective space  $\mathbb{P}^d$ .*

Fano manifolds (or more generally Fano varieties with certain singularities) are important objects since they appear naturally in the minimal model program as the general fibres of Mori fibre spaces. However if one wants to get a more complete picture of Mori fibre spaces, one should also try to obtain some information on the special fibres. In the polarised setting we have a relative version of the Kobayashi–Ochiai theorem:

**1.2. Theorem.** [Fuj87, Lemma 2.12], [Ion86] *Let  $X$  be a manifold and  $\varphi: X \rightarrow Y$  a projective, equidimensional morphism of relative dimension  $d$  onto a normal variety. Suppose that the general fibre  $F$  is isomorphic to  $\mathbb{P}^d$  and that there exists a  $\varphi$ -ample Cartier divisor  $A$  on  $X$  such that the restriction to  $F$  is isomorphic to the hyperplane divisor  $H$ . Then  $\varphi$  is a projective bundle and  $A$  is a global hyperplane divisor.*

---

*Date:* January 20, 2012.

*2000 Mathematics Subject Classification.* 14E30, 14D06, 14J40, 14J45.

*Key words and phrases.* Mori contraction, length of extremal rays, rational curves, degenerations of projective spaces.

A.H. was partially supported by the A.N.R. project “CLASS”.

The existence of the  $\varphi$ -ample Cartier divisor  $A$  being a rather restrictive condition, the object of this paper is to replace it by a more flexible “numerical” hypothesis, i.e. we prove a relative version of Cho–Miyaoka–Shepherd-Barron theorem, even for fibrations that are not equidimensional.

**1.B. Main results.** Let  $X$  be a quasi-projective manifold and let  $\varphi: X \rightarrow Y$  be an *elementary contraction*, i.e. a Mori contraction associated with an extremal ray  $R$  of  $X$ . The *length* of the extremal ray  $R$  (or length of the elementary contraction) is defined as

$$l(R) := \min\{K_X^* \cdot C \mid C \subset X \text{ a rational curve s.t. } [C] \in R\}.$$

Denote by  $E \subset X$  an irreducible component of the  $\varphi$ -exceptional locus ( $E = X$  for a contraction of fibre type), and let  $F$  be an irreducible component of a  $\varphi$ -fibre contained in  $E$ . Then by the Ionescu–Wiśniewski inequality [Ion86, Thm.0.4], [Wiś91a, Thm.1.1] one has

$$(1) \quad \dim E + \dim F \geq \dim X + l(R) - 1.$$

In this paper we investigate the case when (1) is an equality: if the contraction  $\varphi$  maps  $X$  onto a point, Theorem 1.1 says that  $X = E = F$  is a projective space; more generally, Andreatta and Wiśniewski conjectured that  $F$  is a projective space regardless of the dimension of the target [AW97, Conj.2.6]. We prove a strong version of this conjecture for Mori fibre spaces:

**1.3. Theorem.** *Let  $X$  be a quasi-projective manifold that admits an elementary contraction of fibre type  $\varphi: X \rightarrow Y$  onto a normal variety  $Y$  such that the general fibre has dimension  $d$ . Suppose that the contraction has length  $l(R) = d + 1$ .*

*If  $\varphi$  is equidimensional, it is a projective bundle. If  $\varphi$  is not equidimensional, there exists a commutative diagram*

$$\begin{array}{ccc} X' & \xrightarrow{\mu'} & X \\ \varphi' \downarrow & & \downarrow \varphi \\ Y' & \xrightarrow{\mu} & Y \end{array}$$

*such that  $\mu$  and  $\mu'$  are birational,  $X'$  and  $Y'$  are smooth, and  $\varphi': X' \rightarrow Y'$  is a projective bundle.*

Since  $X$  is smooth, Theorem 1.1 immediately implies that a general  $\varphi$ -fibre  $F$  is isomorphic to  $\mathbb{P}^d$ . Our contribution is to prove that the condition on the length severely limits the possible degenerations of these projective spaces. The smoothness of  $X$  is not essential for these degeneration results (cf. Section 3), so one can easily derive analogues of Theorem 1.3 making some assumption on the singularities of the general fibre (e.g. isolated LCIQ singularities [CT07]).

If the contraction  $\varphi$  is birational, the situation is more complicated: if  $E \subset X$  is an irreducible component of the exceptional locus such that for a general fibre  $F$  of  $E \rightarrow \varphi(E) =: Z$  the inequality (1) is an equality, it is not hard to see ([CMSB02, Rem.12], cf. Remark 2.2) that the general fibre is normalised by a finite union of projective spaces. Let  $\tilde{E} \rightarrow E$  be the normalisation and  $\tilde{\varphi}: \tilde{E} \rightarrow \tilde{Z}$  be the fibration obtained by the Stein factorisation of  $\tilde{E} \rightarrow E \rightarrow Z$ . The general  $\tilde{\varphi}$ -fibre is a projective space and we obtain the following:

**1.4. Theorem.** *In the situation above, the fibration  $\tilde{\varphi}: \tilde{E} \rightarrow \tilde{Z}$  is a projective bundle in codimension one. Moreover there exists a commutative diagram*

$$\begin{array}{ccc} E' & \xrightarrow{\mu'} & \tilde{E} \\ \varphi' \downarrow & & \downarrow \tilde{\varphi} \\ Z' & \xrightarrow{\mu} & \tilde{Z} \end{array}$$

*such that  $\mu$  and  $\mu'$  are birational,  $E'$  and  $Z'$  are smooth, and  $\varphi': E' \rightarrow Z'$  is a projective bundle.*

While this result gives a rather precise description of the normalisation  $\tilde{E}$ , it does not prove that the fibre  $F$  itself is a projective space. However, if the contraction  $\varphi$  is divisorial (so the inequality (1) simplifies to  $\dim F \geq l(R)$ ), Andreatta and Occhetta [AO02, Thm.5.1] proved that all the nontrivial fibres of  $\varphi$  have dimension  $l(R)$  if and only if  $X$  is the blow-up of a manifold along a submanifolds of codimension  $l(R)+1$ . If one studies the proof of Kawamata's classification of smooth fourfold flips [Kaw89, Thm.1.1] (which corresponds to the case  $\dim X = 4, \dim E = \dim F = 2, l(R) = 1$ ) one sees that for flipping contractions the proof of the normality requires completely different techniques. We leave this interesting problem for future research.

**1.C. Further developments.** Theorem 1.3 completely determines the equidimensional fibre type contractions of maximal length: they are projective bundles. If the fibration is not equidimensional it still gives a precise description of the Chow family defined by the fibration. It seems reasonable that this description allows to deduce some information about higher-dimensional fibres. For example if  $X$  of dimension  $n$  maps onto a threefold  $Y$ , it is not hard to see that any fibre component of dimension  $n-2$  is normalised by  $\mathbb{P}^{n-2}$ . More generally we expect that the theory of varieties covered by high-dimensional linear spaces [Ein85], [Wiś91b], [BSW92], [ABW92], [Sat97], [NO11] can be applied in this context.

Another interesting line of investigation would be to prove that under additional assumptions the fibration  $\varphi$  is always equidimensional. We recall the following conjecture by Beltrametti and Sommese:

**1.5. Conjecture.** [BS93], [BS95, Conj.14.1.10] *Let  $(X, L)$  be a polarised projective manifold of dimension  $n$  that is an adjunction theoretic scroll  $\varphi: X \rightarrow Y$  over a normal variety  $Y$  of dimension  $m$ . If  $n \geq 2m - 1$ , then  $\varphi$  is equidimensional.*

Wiśniewski [Wiś91b, Thm.2.6] proved this conjecture if  $L$  is very ample and  $n \geq 2m$ , but apart from partial results for low-dimensional  $Y$ , [BSW92], [Som86], [Tir10], this conjecture is very much open.

In Section 5 we use Theorem 1.3 to provide some evidence that a contraction of length  $l(R) = n - m + 1$  is always locally a scroll, so we expect that Conjecture 1.5 even holds in the more general setting of Theorem 1.3.

**1.6. Conjecture.** *Let  $X$  be a projective manifold of dimension  $n$  that admits an elementary contraction of fibre type  $\varphi: X \rightarrow Y$  onto a normal variety  $Y$  of dimension  $m$ . Suppose that the contraction has length  $l(R) = n - m + 1$ . If  $n \geq 2m - 1$ , then  $\varphi$  is equidimensional.*

**Acknowledgements.** We thank C. Araujo for sending us her inspiring preprint [Ara09]. We thank M. Andreatta, S. Druel, H. Hamm, F. Han, P. Popescu-Pampu, M. Wolff and J.A. Wiśniewski for helpful communications.

## 2. NOTATION AND BASIC RESULTS

We work over the complex field  $\mathbb{C}$ . A fibration is a projective surjective morphism  $\varphi: X \rightarrow Y$  with connected fibres between normal varieties such that  $\dim X > \dim Y$ . The  $\varphi$ -equidimensional (resp.  $\varphi$ -smooth) locus is the largest Zariski open subset  $Y^* \subset Y$  such that for every  $y \in Y^*$ , the fibre  $\varphi^{-1}(y)$  has dimension  $\dim X - \dim Y$  (resp. has dimension  $\dim X - \dim Y$  and is smooth).

An elementary Mori contraction of a quasi-projective manifold  $X$  is a morphism with connected fibres  $\varphi: X \rightarrow Y$  onto a normal variety  $Y$  such that the anticanonical divisor  $K_X^*$  is  $\varphi$ -ample and the numerical classes of curves contracted by  $\varphi$  lie on an extremal ray  $R \subset N_1(X)$ .

We will use the notation of [Kol96, Ch.II]: if  $U \rightarrow V$  is a variety  $U$  that is projective over some base  $V$ , we denote by  $\text{RatCurves}^n(U/V)$  the space parameterising rational curves on  $U$ . If moreover  $s: V \rightarrow U$  is a section, we denote by  $\text{RatCurves}^n(s, U/V)$  the space parameterising rational curves passing through  $s(V)$ . In the case where  $V$  is a point and  $x = s(V)$ , we simply write  $\text{RatCurves}^n(U)$  and  $\text{RatCurves}^n(x, U)$ .

Let  $F$  be a normal, projective variety of dimension  $d$ . If  $\mathcal{H} \subset \text{RatCurves}^n(F)$  is an irreducible component parameterising a family of rational curves that dominates  $F$ , then for a general point  $x \in F$  one has

$$\dim \mathcal{H}_x = \dim \mathcal{H} + 1 - d,$$

where  $\mathcal{H}_x \subset \text{RatCurves}^n(x, F)$  parameterises the members of  $\mathcal{H}$  passing through  $x$ . If  $\mathcal{H}_x$  is proper, it follows from bend-and-break [Mor79] that  $\dim \mathcal{H}_x \leq d - 1$ . Thus in this case we have

$$(2) \quad \dim \mathcal{H} \leq 2d - 2.$$

Fix now an ample  $\mathbb{Q}$ -Cartier divisor  $A$  on  $F$ . Let  $C \subset F$  be a rational curve such that

$$A \cdot C = \min\{A \cdot C' \mid C' \subset F \text{ is a rational curve}\};$$

then any irreducible component of  $\mathcal{H} \subset \text{RatCurves}^n(F)$  containing  $C$  is proper [Kol96, II., Prop.2.14], so it parameterises a family of rational curves  $\mathcal{H}$  that is unsplit in the sense of [CMSB02, Defn.0.2]. If moreover the irreducible component  $\mathcal{H}$  has the maximal dimension  $2d - 2$ , the family is doubly dominant, i.e. for every  $x, y \in F$  there exists a member of the family  $\mathcal{H}$  that joins  $x$  and  $y$ . In this setting one can prove a version of Theorem 1.1 that includes normal varieties:

**2.1. Theorem.** [CMSB02, Main Thm.0.1]<sup>1</sup> *Let  $F$  be a normal, projective variety of dimension  $d$  and  $A$  an ample  $\mathbb{Q}$ -Cartier divisor on  $F$ . Suppose that there exists an irreducible component  $\mathcal{H} \subset \text{RatCurves}^n(F)$  of dimension at least  $2d - 2$  such that for a curve  $[C] \in \mathcal{H}$  we have*

$$A \cdot C = \min\{A \cdot C' \mid C' \subset F \text{ is a rational curve}\}.$$

---

<sup>1</sup>A proof of this statement following the strategy in [Keb02] can be found in [CT07, §4].

Then  $F$  is isomorphic to a projective space  $\mathbb{P}^d$  and  $\mathcal{H}$  parameterises the family of lines on  $\mathbb{P}^d$ .

**2.2. Remark.** If  $F$  is not smooth, it is in general quite hard to verify the condition  $\dim \mathcal{H} \geq 2d - 2$  (cf. [CT07]). Since we work with an ambient space that is smooth, things are much simpler:

Let  $X$  be a quasi-projective manifold, and let  $\varphi: X \rightarrow Y$  be a birational elementary contraction. Let  $E \subset X$  be an irreducible component of the  $\varphi$ -exceptional locus such that for an irreducible component  $F$  of a general fibre of  $E \rightarrow \varphi(E)$  the inequality (1) is an equality. We want to describe the structure of  $F$ : taking general hyperplane sections on  $Y$  we can suppose that  $E = F$ , so the boundary case of (1) simplifies to

$$2 \dim F = \dim X + l(R) - 1.$$

Let  $C \subset F \subset X$  be a rational curve passing through a general point  $x \in F$  that has minimal degree with respect to  $K_X^*$ . Then  $C$  belongs to an irreducible family of rational curves  $\mathcal{H} \subset \text{RatCurves}^n(F)$  that dominates  $F$ , moreover  $\mathcal{H}_x$  is proper. Note now that any deformation of  $C$  in  $X$  is contracted by  $\varphi$ , hence contained in  $F$ . Thus we can estimate  $\dim \mathcal{H}$  by applying Riemann–Roch on the *manifold*  $X$ :

$$\dim \mathcal{H} \geq K_X^* \cdot C - 3 + \dim X \geq l(R) - 3 + \dim X = 2 \dim F - 2.$$

By (2) we see that these inequalities are in fact equalities, in particular one has  $K_X^* \cdot C = l(R)$ . Thus if  $\nu: \tilde{F} \rightarrow F$  is the normalisation and  $\tilde{\mathcal{H}} \subset \text{RatCurves}^n(\tilde{F})$  the family obtained by lifting the members of  $\mathcal{H}$ , it satisfies the conditions of Theorem 2.1 with respect to the polarisation  $A := \nu^* K_X^*$ . Thus we have  $\tilde{F} \simeq \mathbb{P}^{\dim F}$  and  $\nu^* K_X^* \equiv l(R)H$  with  $H$  the hyperplane divisor.

### 3. DEGENERATIONS OF $\mathbb{P}^d$

In this section we prove our main results on degenerations of  $\mathbb{P}^d$  satisfying a length condition.

**3.1. Proposition.** *Let  $X$  be a normal, quasi-projective variety and let  $\varphi: X \rightarrow Y$  be an equidimensional fibration of relative dimension  $d$  onto a normal variety such that the general fibre  $F$  is isomorphic to  $\mathbb{P}^d$ . Let  $A$  be a  $\varphi$ -ample  $\mathbb{Q}$ -Cartier divisor, and let  $e \in \mathbb{N}$  be such that  $A|_F \equiv eH$  with  $H$  the hyperplane divisor. Suppose that the following length condition holds:*

$$(3) \quad A \cdot C \geq e \quad \forall C \subset X \text{ rational curve s.t. } \varphi(C) = pt.$$

*Then all the fibres are irreducible and generically reduced. Moreover the normalisation of any fibre is a projective space.*

**3.2. Remark.** In general the degeneration behaviour of projective spaces can be quite complicated, for example it depends in a subtle manner on the geometry of the total space. Consider  $\varphi: X \rightarrow C$  a fibration from a normal variety  $X$  onto a smooth curve  $C$  such that all the fibres are integral and the general fibre is isomorphic to  $\mathbb{P}^d$ . If  $X$  is smooth (or at least factorial), then Tsen's theorem implies that  $X \rightarrow C$  is a  $\mathbb{P}^d$ -bundle ([NO07, Lemma 2.17], cf. also the proof of [DP10, Lemma 4]). This assumption can not be weakened:

**3.3. Example.** [Ara09] Let  $W_4 \subset \mathbb{P}^6$  be the cone over the Veronese surface  $R_4 \subset \mathbb{P}^5$  (that is the 2-uple embedding of  $\mathbb{P}^2$  in  $\mathbb{P}^5$ ). The blow-up of  $W_4$  in the vertex  $P$  is isomorphic to the projectivised bundle  $\mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-2)) \rightarrow \mathbb{P}^2$ ; this desingularisation contracts a  $\mathbb{P}^2$  with normal bundle  $\mathcal{O}_{\mathbb{P}^2}(-2)$  onto the vertex  $P$ . In particular  $W_4$  is a terminal  $\mathbb{Q}$ -factorial threefold and the canonical divisor  $K_{W_4}$  is not Gorenstein, but 2-Gorenstein.

The base locus of a general pencil of hyperplane sections of  $W_4 \subset \mathbb{P}^6$  identifies to a smooth quartic curve  $C \subset R_4$ . If we denote by  $\mu: X \rightarrow W_4$  the blow-up in  $C$ , it is a terminal,  $\mathbb{Q}$ -factorial, 2-Gorenstein threefold admitting a fibration  $\varphi: X \rightarrow \mathbb{P}^1$  such that the fibres are isomorphic to the members of the general pencil, in particular they are integral. Thus the general fibre  $F$  is isomorphic to  $R_4 \simeq \mathbb{P}^2$ , but the fibre  $F_0$  corresponding to the hyperplane section through the vertex  $P$  is a cone over the quartic curve  $C$ . In particular it is not normalised by  $\mathbb{P}^2$ .

Let us note that  $A := K_X \otimes \mu^* \mathcal{O}_{W_4}(2)$  is a  $\varphi$ -ample  $\mathbb{Q}$ -Cartier divisor such that the restriction to a general fibre is numerically equivalent to the hyperplane divisor  $H \subset \mathbb{P}^2$ . However this divisor is not Cartier, so Theorem 1.2 does not apply. Let us also note that under the 2-uple embedding  $\mathbb{P}^2 \hookrightarrow \mathbb{P}^5$ , a line is mapped onto a conic. If we degenerate the general fibre  $F$  to  $F_0$ , these conics degenerate to a union of two lines  $l_1 \cup l_2$  passing through the vertex of the cone  $F_0$ . Since we have  $A \cdot l_i = \frac{1}{2}$ , the length condition (3) is not satisfied.

*Proof of Proposition 3.1.* By assumption the general fibre is a projective space. Let  $\mathcal{H} \subset \text{RatCurves}^n(X/Y)$  be the unique irreducible component such that a general point corresponds to a line  $l$  contained in the general fibre  $F$ . We have  $A \cdot l = e$  and by (3) one has  $A \cdot C \geq e$  for every rational curve  $C$  contained in a fibre, so the variety  $\mathcal{H}$  is proper over the base  $Y$  [Kol96, II., Prop. 2.14]. Since the general fibre of  $\mathcal{H} \rightarrow Y$  corresponds to the  $2d - 2$ -dimensional family of lines in the projective space  $\mathbb{P}^d \simeq F$ , it follows by upper semicontinuity that for every  $0 \in Y$ , all the irreducible components of the fibre  $\mathcal{H}_0$  have dimension at least  $2d - 2$ . Let  $\mathcal{H}_{0,i}$  be the normalisation of such an irreducible component and  $\mathcal{U}_{0,i} \rightarrow \mathcal{H}_{0,i}$  be the universal family over it. The image of the evaluation morphism  $p: \mathcal{U}_{0,i} \rightarrow X$  is an irreducible component  $D_i$  of the set-theoretical fibre  $(\varphi^{-1}(0))_{\text{red}}$ , so it has dimension  $d$ . Let  $\nu: \tilde{D}_i \rightarrow D_i$  be the normalisation, then the family of rational curves  $\mathcal{H}_{0,i}$  lifts to  $\tilde{D}_i$ , so  $\text{RatCurves}^n(\tilde{D}_i)$  has an irreducible component of dimension at least  $2d - 2$ . Since for any rational curve  $C \subset D_i \subset X$  we have  $A \cdot C \geq e$  by (3), the pull-back  $\nu^* A$  also satisfies this inequality. Moreover  $\nu^* A$  has degree exactly  $e$  on the rational curves parameterised by  $\mathcal{H}_{0,i}$ . Thus we conclude with Theorem 2.1 that  $\tilde{D}_i$  is isomorphic to a projective space.

We argue now by contradiction and suppose that there exists a  $0 \in Y$  such that the fibre  $F_0 := \varphi^{-1}(0)$  is reducible or not generically reduced. Then we can decompose the cycle

$$[F_0] = \sum_i a_i [D_i]$$

with  $a_i \in \mathbb{N}$  and  $D_i$  the irreducible components of the set-theoretical fibre  $(\varphi^{-1}(0))_{\text{red}}$ . Since the degree is constant in a well-defined family of proper algebraic cycles [Kol96, Prop.3.12], we have

$$e^d = [F] \cdot A^d = [F_0] \cdot A^d = \sum_i a_i ([D_i] \cdot A^d).$$

By assumption the sum on the right hand side is not trivial, so we have  $[D_1] \cdot A^d < e^d$ . Denote by  $\nu: \mathbb{P}^d \rightarrow D_1$  the normalisation, then we have

$$(\nu^* A)^d = [D_1] \cdot A^d < e^d.$$

Thus we see that  $\nu^* A \equiv bH$  with  $0 < b < e$ . Yet this is impossible, since it implies that for a line  $l \subset \mathbb{P}^d$ , we have

$$A \cdot \nu(l) = \nu^* A \cdot l = b < e,$$

a contradiction to (3).  $\square$

**3.4. Lemma.** *Let  $\varphi: X \rightarrow Y$  be an equidimensional fibration from a normal variety  $X$  onto a manifold  $Y$ . Suppose that  $\varphi$  is generically smooth and the general fibre  $F$  is a Fano manifold with Picard number one. If  $\varphi$  has irreducible and generically reduced fibres, then  $X$  is  $\mathbb{Q}$ -Gorenstein.*

*Proof.* Denote by  $H$  the ample generator of  $\text{Pic}(F)$  and let  $A$  be a  $\varphi$ -ample Cartier divisor. Set  $e \in \mathbb{N}$  such that  $A|_F \equiv eH$ , and  $d \in \mathbb{N}$  such that  $K_F^* \equiv dH$ . The reflexive sheaf  $\mathcal{O}_X(eK_X + dA)$  is locally free in a neighborhood of a general fibre  $F$  and the restriction  $\mathcal{O}_X(eK_X + dA) \otimes \mathcal{O}_F$  is isomorphic to the structure sheaf  $\mathcal{O}_F$ . Up to replacing  $A$  by  $A \otimes \varphi^* H$  with  $H$  a sufficiently ample Cartier divisor on  $Y$  we can suppose without loss of generality that

$$H^0(X, \mathcal{O}_X(eK_X + dA)) \simeq H^0(Y, \varphi_* \mathcal{O}_X(eK_X + dA)) \neq 0.$$

Thus we obtain a non-zero morphism

$$\mathcal{O}_X(eK_X^*) \rightarrow \mathcal{O}_X(dA)$$

which is an isomorphism on the general fibre  $F$ . The vanishing locus is thus a Weil divisor  $\sum k_i D_i$  such that for all  $i$  we have  $\varphi(D_i) \subsetneq X$ . Since  $\varphi$  is equidimensional and all the fibres are irreducible and generically reduced we see that  $D_i = \varphi^* E_i$  with  $E_i$  a prime divisor on  $Y$ . Since  $Y$  is smooth, the divisor  $E_i$  is Cartier, hence  $\sum k_i D_i = \sum k_i \varphi^* E_i$  is Cartier. Thus we have an isomorphism

$$\mathcal{O}_X(eK_X^*) \simeq \mathcal{O}_X(dA - \sum k_i \varphi^* E_i).$$

The right hand side is Cartier, so  $K_X$  is  $\mathbb{Q}$ -Cartier.  $\square$

**3.5. Proposition.** *In the situation of Proposition 3.1, suppose that  $Y$  is smooth. Then the fibration  $\varphi$  is a projective bundle.*

*Proof.* We will proceed by induction on the relative dimension  $D$ , the case  $d = 1$  is [Kol96, II, Thm.2.8]. Taking general hyperplane sections on  $Y$  and arguing by induction on the dimension we can suppose without loss of generality that  $\varphi$  has at most finitely many singular fibres. The problem being local we can suppose that  $Y \subset \mathbb{C}^{\dim Y}$  is a polydisc around 0 and  $\varphi$  is smooth over  $Y \setminus 0$ . Since all the fibres are generically reduced by Proposition 3.1 we can choose a section  $s: Y \rightarrow X$  such that  $s(0) \subset F_{0, \text{non-s}}$  where  $F_0 := \varphi^{-1}(0)$ .

As in the proof of Proposition 3.1 we denote by  $\mathcal{H} \subset \text{RatCurves}^n(X/Y)$  the unique irreducible component such that a general point corresponds to a line  $l$  contained in the general fibre  $F \simeq \mathbb{P}^d$ . There exists a unique irreducible component  $\mathcal{H}_s \subset \text{RatCurves}^n(s, X/Y)$  such that for general  $y \in Y$  the lines in  $\varphi^{-1}(y) \simeq \mathbb{P}^d$  passing through  $s(y)$  are parameterised by  $\mathcal{H}_s$ . If we denote by  $\psi: \mathcal{H}_s \rightarrow Y$  the natural fibration, its general fibre is isomorphic to a projective space  $\mathbb{P}^{d-1}$ . Let  $q: \mathcal{U}_s \rightarrow \mathcal{H}_s$  be the universal family, then by [Kol96, Ch.II, Cor.2.12] the fibration  $q$  is a  $\mathbb{P}^1$ -bundle. Let  $p: \mathcal{U}_s \rightarrow X$  be the evaluation morphism, then  $p$  is birational since this holds for the restriction to a general  $\varphi$ -fibre. The variety  $X$  being normal, we know by Zariski's main theorem that  $p$  has connected fibres. The family of rational curves being unsplit, it follows from bend-and-break that  $p$  has finite fibres over  $X \setminus s(Y)$ . Thus we have an isomorphism

$$(4) \quad X \setminus s(Y) \simeq \mathcal{U}_s \setminus E,$$

where  $E := (p^{-1}(s(Y)))_{\text{red}}$ . Again by bend-and-break and connectedness of the fibres the algebraic set  $E$  is irreducible, so it is a prime divisor with a finite, birational morphism  $q|_E: E \rightarrow \mathcal{H}_s$ . Since  $\mathcal{H}_s$  is normal, we see by Zariski's main theorem that  $q|_E$  is an isomorphism and  $E$  is a  $q$ -section.

By Lemma 3.4 we know that  $X$  is  $\mathbb{Q}$ -Gorenstein, so  $\mathcal{U}_s \setminus E$  is  $\mathbb{Q}$ -Gorenstein. Since  $(\mathcal{U}_s \setminus E) \rightarrow \mathcal{H}_s$  is locally trivial (it is a  $\mathbb{C}$ -bundle), it follows that  $\mathcal{H}_s$  is  $\mathbb{Q}$ -Gorenstein. Since  $\mathcal{U}_s \rightarrow \mathcal{H}_s$  is a  $\mathbb{P}^1$ -bundle, it now follows that the total space  $\mathcal{U}_s$  is  $\mathbb{Q}$ -Gorenstein. The  $\mathbb{P}^1$ -bundle  $\mathcal{U}_s \rightarrow \mathcal{H}_s$  has a section  $E$ , so it is isomorphic to the projectivised bundle  $\mathbb{P}(V) \rightarrow \mathcal{H}_s$  with  $V := \varphi_* \mathcal{O}_X(E)$ . By the canonical bundle formula we have

$$(5) \quad K_{\mathcal{U}_s} \equiv q^*(K_{\mathcal{H}_s} + \det V) - 2E.$$

We claim that the following length condition holds:

$$(6) \quad K_{\mathcal{H}_s}^* \cdot C \geq d \quad \forall C \subset \mathcal{H}_s \text{ rational curve s.t. } \psi(C) = pt.$$

Assuming this for the time being, let us see how to conclude: applying the induction hypothesis to  $\psi$ , we see that  $\mathcal{H}_s \rightarrow Y$  is a  $\mathbb{P}^{d-1}$ -bundle. Since  $q$  is a  $\mathbb{P}^1$ -bundle, we see that the central fibre of  $\mathcal{U}_s \xrightarrow{\psi \circ q} Y$  is a  $\mathbb{P}^1$ -bundle over  $\mathbb{P}^{d-1}$  that contracts a section onto a smooth point of  $F_0$ . A classical argument [Mor79] shows that  $F_0 \simeq \mathbb{P}^d$ .

*Proof of the claim.* The claim is obvious for curves in the general fibres which are isomorphic to  $\mathbb{P}^{d-1}$ , so we can concentrate on curves in the central fibre  $\mathcal{H}_0 := (\psi^{-1}(0))_{\text{red}}$ . We set  $\mathcal{U}_0 := q^{-1}(\mathcal{H}_0)$  and denote by  $\nu: \tilde{\mathcal{H}}_0 \rightarrow \mathcal{H}_0$  the normalisation. Then  $\mathcal{U}_0 \times_{\mathcal{H}_0} \tilde{\mathcal{H}}_0$  is normal and  $\tilde{q}_0: \mathcal{U}_0 \times_{\mathcal{H}_0} \tilde{\mathcal{H}}_0 \rightarrow \tilde{\mathcal{H}}_0$  is a  $\mathbb{P}^1$ -bundle. By Proposition 3.1 we already know that  $F_{0,\text{red}}$  is normalised by  $\mathbb{P}^d$ , so we obtain a commutative diagram:

$$\begin{array}{ccccc}
 & & & & \mathbb{P}^d \\
 & & & \nearrow \mu & \downarrow \\
 \mathcal{U}_0 \times_{\mathcal{H}_0} \tilde{\mathcal{H}}_0 & \xrightarrow{\tilde{\nu}} & \mathcal{U}_0 & \xrightarrow{p_0} & F_{0,\text{red}} \\
 \downarrow \tilde{q} & & \downarrow q_0 & & \\
 \tilde{\mathcal{H}}_0 & \xrightarrow{\nu} & \mathcal{H}_0 & & 
 \end{array}$$



Thus the curves parameterised by  $\tilde{\mathcal{H}}_0$  are lines in  $\mathbb{P}^d$ , in particular we have  $\tilde{\mathcal{H}}_0 \simeq \mathbb{P}^{d-1}$  and

$$\mathcal{U}_0 \times_{\mathcal{H}_0} \tilde{\mathcal{H}}_0 \simeq \mathbb{P}(\mathcal{O}_{\mathbb{P}^{d-1}} \oplus \mathcal{O}_{\mathbb{P}^{d-1}}(-1))$$

with  $\tilde{\nu}^{-1}(E \cap \mathcal{U}_0)$  corresponding to the exceptional section.

In order to check the length condition (6) it is sufficient to do this for a curve  $C_0 \subset \mathcal{H}_0$  such that  $C_0 = \nu(l_0)$  with  $l_0$  a general line in  $\tilde{\mathcal{H}}_0 \simeq \mathbb{P}^{d-1}$ . We can lift  $l_0$  to a curve  $l' \subset \mathbb{P}(\mathcal{O}_{\mathbb{P}^{d-1}} \oplus \mathcal{O}_{\mathbb{P}^{d-1}}(-1))$  that is disjoint from the exceptional section, so we obtain a curve  $C' := \tilde{\nu}(l') \subset \mathcal{U}_0 \setminus E$  such that  $C' \rightarrow C_0$  is birational. In particular we have

$$K_{\mathcal{H}_s}^* \cdot C_0 = q^* K_{\mathcal{H}_s}^* \cdot C',$$

so it is sufficient to show that the right hand side is equal to  $d$ . The rational curve  $C'$  does not meet  $E$  and  $\mu(l')$  is a line, so  $C'$  corresponds to a point in  $\mathcal{H}$  that is not in  $\mathcal{H}_s$ . Using the isomorphism (4) we can deform  $C'$  to a curve  $C'' \subset \mathcal{U}_s$  that is a line in a general fibre  $\varphi^{-1}(y) \simeq \mathbb{P}^d$  and which does not meet the point  $s(y)$ , so  $C''$  is disjoint from  $E$ . Yet for such a curve the Formula (5) restricted to a general fibre immediately shows that  $q^* K_{\mathcal{H}_s}^* \cdot C'' = d$ .  $\square$

**3.6. Remark.** Proposition 3.5 should be true without the assumption that  $Y$  is smooth. In fact this assumption is only needed to assure via Lemma 3.4 that  $X$  (and by consequence  $\mathcal{H}_s$  and  $\mathcal{U}_s$ ) are  $\mathbb{Q}$ -Gorenstein. However our computations only use that these varieties are “relatively  $\mathbb{Q}$ -Gorenstein”, i.e. some multiple of the canonical divisor is linearly equivalent to a Cartier divisor plus some Weil divisors that are pull-backs from the base  $Y$ . We leave the technical details to the interested reader.

#### 4. PROOFS OF MAIN RESULTS

*Proof of Theorem 1.3.* By Theorem 1.1 the general  $\varphi$ -fibre is a projective space.

If  $\varphi$  is equidimensional, then  $A := K_X^*$  satisfies the length condition (3) in Proposition 3.1 with  $e = l(R)$ . Thus we know that every fibre is generically reduced. In particular for every  $y \in Y$  there exists a point  $x \in \varphi^{-1}(y)$  such that the tangent map has rank  $\dim Y$  in  $x$ . Since  $X$  is smooth, this implies that  $Y$  is smooth in  $y$ . Conclude with Proposition 3.5.

Suppose now that  $\varphi$  is not equidimensional. Let now  $\bar{Y}$  be the closure of the  $\varphi$ -equidimensional locus in the Chow scheme  $\mathcal{C}(X)$  and let  $\bar{X} \rightarrow \bar{Y}$  be the universal family. Let  $Y' \rightarrow \bar{Y}$  be a desingularisation, and set  $X'$  for the normalisation of  $\bar{X} \times_{\bar{Y}} Y'$ . We denote by  $\varphi': X' \rightarrow Y'$  the natural fibre space structure and by  $\mu': X' \rightarrow X$  the birational morphism induced by the map  $\bar{X} \rightarrow X$ . By the rigidity lemma there exists a birational morphism  $\mu: Y' \rightarrow Y$  such that  $\mu \circ \varphi' = \varphi \circ \mu'$ . Note also that the restriction of  $\mu'$  to any  $\varphi'$ -fibre is finite, so the pull-back  $A := (\mu')^* K_X^*$  is a  $\varphi'$ -ample Cartier divisor. The divisor  $A$  on  $X'$  satisfies the length condition (3) in Proposition 3.1 with  $e = l(R)$ . Since  $Y'$  is smooth, we can conclude with the Proposition 3.5.  $\square$

*Proof of Theorem 1.4.* By Remark 2.2 the general  $\tilde{\varphi}$ -fibre is a projective space. Moreover if  $\nu: \tilde{E} \rightarrow E \subset X$  denotes the normalisation, then  $A := \nu^* K_X^*$  (restricted to the  $\tilde{\varphi}$ -equidimensional locus) satisfies the length condition (3) in Proposition 3.1 with  $e = l(R)$ . If  $C \subset \tilde{Z}$  is a curve cut out by general hyperplane sections, then  $C$

is smooth and the fibration  $\tilde{\varphi}^{-1}(C) \rightarrow C$  is equidimensional, so it is a projective bundle by Proposition 3.5. Thus  $\tilde{\varphi}$  is a projective bundle in codimension one.

The proof of the second statement is analogous to the proof of the second statement in Theorem 1.3.  $\square$

## 5. LOCAL SCROLL STRUCTURES

We will use the following local version of [BS95, Defn.3.3.1]:

**5.1. Definition.** *Let  $X$  be a manifold, and let  $\varphi: X \rightarrow Y$  be a fibration onto a normal variety  $Y$ . We say that  $\varphi$  is locally a scroll if for every  $y \in Y$  there exists an analytic neighborhood  $y \in U \subset Y$  such that on  $X_U := \varphi^{-1}(U)$  there exists a relatively ample Cartier divisor  $L$  such that the general fibre  $F$  polarised by  $L|_F$  is isomorphic to  $(\mathbb{P}^d, H)$ .*

Suppose now that we are in the situation of Theorem 1.3 and denote by  $Y_0 \subset Y$  the  $\varphi$ -equidimensional locus. Then  $Y_0$  is also the  $\varphi$ -smooth locus by Theorem 1.3. Note also that  $Y$  is factorial and has at most rational singularities [Kol86], so it has at most canonical singularities [KM98, Cor.5.24]. Thus if  $\mu: Y' \rightarrow Y$  is the resolution of singularities in the statement of the theorem, we know by [Tak03] that for any point  $y \in Y$  there exists a contractible analytic neighborhood  $U \subset Y$  such that  $\mu^{-1}(U)$  is simply connected. Thus the local systems  $\varphi'_*\mathbb{Z}_{X'}$  and  $R^2\varphi'_*\mathbb{Z}_{X'}$  are trivial on  $\mu^{-1}(U)$ . Since the restriction of these local systems to  $U_0 := Y_0 \cap U$  coincides with  $(\varphi_*\mathbb{Z}_X)_{U_0}$  and  $(R^2\varphi_*\mathbb{Z}_X)_{U_0}$ , the latter are trivial on  $U_0$ .

**5.2. Lemma.** *In the situation of Theorem 1.3, using the notation above, suppose that  $H^3(U_0, \mathbb{Z})$  is torsion-free. Then  $\varphi$  is a scroll over  $U$ .*

*Proof.* We follow the argument in [AM97, Lemma 3.3], [Ara09]. Set  $X_0 := \varphi^{-1}(U_0)$ . We use the Leray spectral sequence for the smooth morphism  $\varphi: X_0 \rightarrow U_0$ , so set  $E_2^{p,q} = H^p(U_0, R^q\varphi_*\mathbb{Z}_{X_0})$ . We have  $R^1\varphi_*\mathbb{Z}_{X_0} = 0$ , so  $E_2^{p,1} = 0$  for all  $p$ . By what precedes we have  $\varphi_*\mathbb{Z}_{X_0} \simeq \mathbb{Z}_{U_0}$  and  $R^2\varphi_*\mathbb{Z}_{X_0} \simeq \mathbb{Z}_{U_0}$ . Since the fibre of  $R^2\varphi_*\mathbb{Z}_{X_0}$  in any point  $y \in U_0$  identifies to  $H^2(X_y, \mathbb{Z}) \simeq \text{Pic}(\mathbb{P}^d) \simeq \mathbb{Z}[H]$ , we have

$$E_2^{0,2} \simeq \mathbb{Z}[H].$$

The spectral sequence gives a map  $d_3: E_3^{0,2} \rightarrow E_3^{3,0}$  with kernel  $E_4^{0,2} = E_\infty^{0,2}$ . We have

$$E_3^{0,2} \simeq E_2^{0,2} \simeq \mathbb{Z}[H]$$

and

$$E_3^{3,0} \simeq E_2^{3,0} \simeq H^3(U_0, \mathbb{Z}),$$

so we get an exact sequence

$$0 \rightarrow E_\infty^{0,2} \rightarrow E_2^{0,2} \rightarrow H^3(U_0, \mathbb{Z}).$$

Consider now the composed map

$$r: \text{Pic}(X_0) \rightarrow H^2(X_0, \mathbb{Z}) \rightarrow E_\infty^{0,2} \rightarrow E_2^{0,2} \simeq \mathbb{Z}[H].$$

By the exact sequence above we know that the cokernel of  $r$  injects into  $H^3(U_0, \mathbb{Z})$ . Since  $\varphi$  is projective, the restriction map  $r$  is not zero, so its cokernel is torsion. By hypothesis  $H^3(U_0, \mathbb{Z})$  is torsion-free, so  $r$  is surjective.

Thus there exists a Cartier divisor  $L_0$  on  $X_0$  such that  $L_0|_F \equiv H$ . Since  $\varphi$  is smooth in codimension one and does not contract a divisor, the set  $X_U \setminus X_0$  has codimension at least two in  $X_U := \varphi^{-1}(U)$ . Therefore  $L_0$  extends to a  $\varphi$ -ample Cartier divisor  $L$  on  $X_U$ .  $\square$

**5.3. Remark.** Let us note that if  $\pi_1(U_0) \simeq \pi_2(U_0) \simeq \{1\}$  the technical condition in the preceding lemma is satisfied: by the Hurewicz theorem [Hat02, Thm.4.32] one has  $H_2(U_0, \mathbb{Z}) = 0$ , so  $H^3(U_0, \mathbb{Z})$  is torsion-free by [Hat02, Cor.3.3]. The topological conditions  $\pi_1(U_0) \simeq \pi_2(U_0) = \{1\}$  are known to be satisfied in the following two situations:

- $U$  is smooth<sup>2</sup>; or
- $\dim U \geq 4$  and  $U$  has at most isolated lci singularities [Ham71, Kor.1.3].

If  $\dim U = 3$  Hamm's theorem does not apply to  $\pi_2(U_0)$  and it seems not to be clear if  $H^3(U_0, \mathbb{Z})$  is torsion-free, even for cDV-singularities.

## REFERENCES

- [ABW92] Marco Andreatta, Edoardo Ballico, and Jarosław A. Wiśniewski. Projective manifolds containing large linear subspaces. In *Classification of irregular varieties (Trento, 1990)*, volume 1515 of *Lecture Notes in Math.*, pages 1–11. Springer, Berlin, 1992.
- [AM97] Marco Andreatta and Massimiliano Mella. Contractions on a manifold polarized by an ample vector bundle. *Trans. Amer. Math. Soc.*, 349(11):4669–4683, 1997.
- [AO02] Marco Andreatta and Gianluca Occhetta. Special rays in the Mori cone of a projective variety. *Nagoya Math. J.*, 168:127–137, 2002.
- [Ara09] Carolina Araujo. Flat deformations of  $\mathbb{P}^n$ . *unpublished preprint*, 2009.
- [AW97] Marco Andreatta and Jarosław A. Wiśniewski. A view on contractions of higher-dimensional varieties. In *Algebraic geometry—Santa Cruz 1995*, volume 62 of *Proc. Sympos. Pure Math.*, pages 153–183. Amer. Math. Soc., Providence, RI, 1997.
- [BS93] Mauro C. Beltrametti and Andrew J. Sommese. Comparing the classical and the adjunction-theoretic definition of scrolls. In *Geometry of complex projective varieties (Cetraro, 1990)*, volume 9 of *Sem. Conf.*, pages 55–74. Mediterranean, Rende, 1993.
- [BS95] Mauro C. Beltrametti and Andrew J. Sommese. *The adjunction theory of complex projective varieties*, volume 16 of *de Gruyter Expositions in Mathematics*. Walter de Gruyter & Co., Berlin, 1995.
- [BSW92] Mauro C. Beltrametti, Andrew J. Sommese, and Jarosław A. Wiśniewski. Results on varieties with many lines and their applications to adjunction theory. In *Complex algebraic varieties (Bayreuth, 1990)*, volume 1507 of *Lecture Notes in Math.*, pages 16–38. Springer, Berlin, 1992.
- [CMSB02] Koji Cho, Yoichi Miyaoka, and Nicholas I. Shepherd-Barron. Characterizations of projective space and applications to complex symplectic manifolds. In *Higher dimensional birational geometry (Kyoto, 1997)*, volume 35 of *Adv. Stud. Pure Math.*, pages 1–88. Math. Soc. Japan, Tokyo, 2002.
- [CT07] Jiun-Cheng Chen and Hsian-Hua Tseng. Note on characterization of projective spaces. *Comm. Algebra*, 35(11):3808–3819, 2007.
- [DP10] Stéphane Druel and Matthieu Paris. Characterizations of projective spaces and hyperquadrics. *arxiv preprint*, 1012.5238, 2010.
- [Ein85] Lawrence Ein. Varieties with small dual varieties. II. *Duke Math. J.*, 52(4):895–907, 1985.
- [Fuj87] Takao Fujita. On polarized manifolds whose adjoint bundles are not semipositive. In *Algebraic geometry, Sendai, 1985*, volume 10 of *Adv. Stud. Pure Math.*, pages 167–178. North-Holland, Amsterdam, 1987.
- [Ham71] Helmut Hamm. Lokale topologische Eigenschaften komplexer Räume. *Math. Ann.*, 191:235–252, 1971.

---

<sup>2</sup>Note that  $U \setminus U_0$  has codimension at least three, so  $\pi_i(U) \simeq \pi_i(U_0)$  for  $i = 1, 2$ .

- [Hat02] Allen Hatcher. *Algebraic topology*. Cambridge University Press, Cambridge, 2002.
- [Ion86] Paltin Ionescu. Generalized adjunction and applications. *Math. Proc. Cambridge Philos. Soc.*, 99(3):457–472, 1986.
- [Kaw89] Yujiro Kawamata. Small contractions of four-dimensional algebraic manifolds. *Math. Ann.*, 284(4):595–600, 1989.
- [Keb02] Stefan Kebekus. Characterizing the projective space after Cho, Miyaoka and Shepherd-Barron. In *Complex geometry (Göttingen, 2000)*, pages 147–155. Springer, Berlin, 2002.
- [KM98] János Kollár and Shigefumi Mori. *Birational geometry of algebraic varieties*, volume 134 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1998. With the collaboration of C. H. Clemens and A. Corti.
- [KO73] Shoshichi Kobayashi and Takushiro Ochiai. Characterizations of complex projective spaces and hyperquadrics. *J. Math. Kyoto Univ.*, 13:31–47, 1973.
- [Kol86] János Kollár. Higher direct images of dualizing sheaves. I. *Ann. of Math. (2)*, 123(1):11–42, 1986.
- [Kol96] János Kollár. *Rational curves on algebraic varieties*, volume 32 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics*. Springer-Verlag, Berlin, 1996.
- [Mor79] Shigefumi Mori. Projective manifolds with ample tangent bundles. *Ann. of Math. (2)*, 110(3):593–606, 1979.
- [NO07] Carla Novelli and Gianluca Occhetta. Ruled Fano fivefolds of index two. *Indiana Univ. Math. J.*, 56(1):207–241, 2007.
- [NO11] Carla Novelli and Gianluca Occhetta. Projective manifolds containing a large linear subspace with nef normal bundle. *Michigan Math. J.*, 60:441–462, 2011.
- [Sat97] Eiichi Sato. Projective manifolds swept out by large-dimensional linear spaces. *Tohoku Math. J. (2)*, 49(3):299–321, 1997.
- [Som86] Andrew J. Sommese. On the adjunction theoretic structure of projective varieties. In *Complex analysis and algebraic geometry (Göttingen, 1985)*, volume 1194 of *Lecture Notes in Math.*, pages 175–213. Springer, Berlin, 1986.
- [Tak03] Shigeharu Takayama. Local simple connectedness of resolutions of log-terminal singularities. *Internat. J. Math.*, 14(8):825–836, 2003.
- [Tir10] Andrea L. Tironi. Scrolls over four dimensional varieties. *Adv. Geom.*, 10(1):145–159, 2010.
- [Wiś91a] Jarosław A. Wiśniewski. On contractions of extremal rays of Fano manifolds. *J. Reine Angew. Math.*, 417:141–157, 1991.
- [Wiś91b] Jarosław A. Wiśniewski. On deformation of nef values. *Duke Math. J.*, 64(2):325–332, 1991.

ANDREAS HÖRING, UNIVERSITÉ PIERRE ET MARIE CURIE, INSTITUT DE MATHÉMATIQUES DE JUSSIEU, PROJET TOPOLOGIE ET GÉOMÉTRIE ALGÈBRIQUES, CASE 247, 4 PLACE JUSSIEU, 75005 PARIS, FRANCE

*E-mail address:* hoering@math.jussieu.fr

CARLA NOVELLI, DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ DEGLI STUDI DI PADOVA, VIA TRIESTE 63, I-35121 PADOVA, ITALY

*E-mail address:* novelli@math.unipd.it